Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Symmetric Binary Forms and Involutions.—*Continued.*

By Arthur B. Coble.

---

§ 11. *Introduction.*

This paper is a continuation of two others under the same title in this Journal.* The numbering of paragraphs and theorems follows that of the earlier papers.

As a preliminary to the study of the symmetric form $H_{2,3}$ some facts and formulae which refer to the forms $H_{1,3}$ and $H_{0,3}$ are necessary.

The binary cubic

$$(a\bar{x})^3 = -[a_2^3\bar{x}_1^3 - 3a_2^2a_1\bar{x}_1^2\bar{x}_2 + 3a_2a_1^2\bar{x}_1\bar{x}_2^2 - a_1^3\bar{x}_2^3]$$

will be represented by a point X in a space S_3, and the binary cubic, $(ax)^3$, similarly expanded will be represented by a plane U if we write

$$\begin{array}{llll} X_0 = a_2^3, & X_1 = -3a_2^2a_1, & X_2 = 3a_2a_1^2, & X_3 = -a_1^3; \\ U_0 = a_1^3, & U_1 = a_1^2a_2, & U_2 = a_1a_2^2, & U_3 = a_2^3. \end{array}$$

Evidently point X and plane U determined by the same binary cubic are point and null-plane in the null-system set up by a cubic norm-curve, N_3, in S_3.

The condition that the four points, $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$, or the four planes, $U_{(1)}$, $U_{(2)}$, $U_{(3)}$, $U_{(4)}$, which represent the cubics, $(ax)^3$, $(bx)^3$, $(cx)^3$, $(dx)^3$, shall, respectively, lie in a plane, or lie on a point, is that the four cubics be linearly related. The condition that plane U given by $(ax)^3$ and point X given by $(b\bar{x})^3$ be incident is that the cubics be apolar. The formulae of transition are

$$\begin{aligned} (101) \qquad (X_{(1)}\,X_{(2)}\,X_{(3)}\,X_{(4)}) &= 9\,(ab)(ac)(ad)(bc)(bd)(cd), \\ (U_{(1)}\,U_{(2)}\,U_{(3)}\,U_{(4)}) &= (ab)(ac)(ad)(bc)(bd)(cd), \\ (UX) &= (ab)^3.\dagger \end{aligned}$$

---

* Vol. XXXI, p. 183 and p. 355.

† In writing the binary form which corresponds to (UX) we shall, as a matter of convention, always put first in the apolarity condition the cubic which corresponds to U.

The symmetric form

$$H_{1,3} = (a_1 x_1)^3 (a_2 x_2)^3$$

is represented by a quadric. The point pair, $(a \bar{x})^3$, $(b \bar{x})^3$, is apolar to the quadric if $(a_1 a)^3 (a_2 b)^3 = 0$. The three values of x_2 determined by $H_{1,3}$ when x_1 is given are the parameters on N_3 of the three points in which the polar plane of the point, x_1, on N_3 as to the quadric meets N_3.

From (15) and (16) or directly from (101) we find the binary comitants of the form $H_{1,3}$ which correspond to the quaternary comitants (polarized) of the quadric. The formulae are

$$(102) \qquad \begin{aligned} a_x a_y = a'_x a'_y &= (a_1 x_1)^3 (a_2 x_2)^3 = (a'_1 x_1)^3 (a'_2 x_2)^3 = \ldots\ldots, \\ (a\,a'\,a''\,u)(a\,a'\,a''\,v) &= (a_1 a'_1)(a_1 a''_1)(a'_1 a''_1)(a_2 a'_2)(a_2 a''_2)(a'_2 a''_2) \\ &\qquad (a_1 \bar{x}_1)(a'_1 \bar{x}_1)(a''_1 \bar{x}_1)(a_2 \bar{x}_2)(a'_2 \bar{x}_2)(a''_2 \bar{x}_2) \\ &= (\alpha_1 \bar{x}_1)^3 (\alpha_2 \bar{x}_2)^3, \\ (a\,a'\,a''\,a''')^2 &= (a_1 \alpha_1)^3 (a_2 \alpha_2)^3 \\ &= (a_1 a'_1) \ldots\ldots (a''_1 a'''_1)(a_2 a'_2) \ldots\ldots (a''_2 a'''_2). \end{aligned}$$

Also let

$$N_x N_y = (x_1 x_2)^2 \,.\, (a\, x_1)(a\, x_2)$$

be the ∞^2 quadrics which contain N_3. A member of the system is singled out by the quadratic, $(a x)^2$, which determines the two tangent lines of N_3 which also lie on the quadric.

The apolarity of the quadric $H_{1,3}$ in planes with all the quadrics containing N_3, *i. e.*,

$$(a\,a'\,a''\,N)^2 = (\alpha_1 \alpha_2)^2 (\alpha_1 x)(\alpha_2 x) \equiv 0,$$

has been found in (23) to be the condition that the form $H_{1,3}$ determine an involution $I_{1,3}$. If this condition is satisfied and, at the same time, $(a\,a'\,a''\,a''')^2 = 0$, then $H_{1,3}$ in planes is the square of the double point of $H_{1,3}$, and this double point must lie on N_3. The parameter of the double point is the sixfold root of $(\alpha_1 x)^3 (\alpha_2 x)^3 = 0$. The $I_{1,3}$ consists of an $I_{1,2}$ and a neutral point, the double point itself. In fact the quadric cone $H_{1,3}$ and the quadric cone which projects N_3 from the double point of $H_{1,3}$ are cut by any plane in apolar conics in the involution relation of § 6.

If $H_{1,3}$ is a pair of planes, the condition, $(a\,a'\,a''\,N)^2 = 0$, is satisfied, but it is clear that $H_{1,3}$ will have the involution property only when the axis of the planes cuts N_3 twice. The $I_{1,3}$ is then made up of two points, each neutral, whose parameters are determined by the axis, and an $I_{1,1}$.

§ 12. *The Form $H_{2,3}$ as a Cubic Surface.*

Writing as usual

$$H_{2,3} = (a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3 = (a_1' x_1)^3 (a_2' x_2)^3 (a_3' x_3)^3 = \dots ,$$

we recall the representations of $H_{2,3}$ as a *parametric spread* and an *apolarity spread* with respect to N_3.

If x_1, x_2, x_3 satisfy the relation, $H_{2,3} = 0$, the planes of N_3 with these parameters meet in a point X of the parametric spread which is manifestly a general cubic surface, Γ_3. If $x_1, \dots, x_5$ are an involutive set of $H_{2,3}$, the corresponding five planes of N_3 form "an inscribed 5-plane of Γ_3" (§ 3).

If x_1, x_2, x_3 satisfy the relation, $H_{2,3} = 0$, the points of N_3 with these parameters form an apolar triad of the apolarity spread which is also a general cubic surface, C_3. The point X which represents the cubic $(a\bar{x})^3 = (a'\bar{x})^3 = (a''\bar{x})^3$ lies on C_3 if $(a_1 a)^3 (a_2 a')^3 (a_3 a'')^3 = 0$. If $x_1, \dots, x_5$ are an involutive set of $H_{2,3}$, the corresponding five points of N_3 form "an antorthic 5-point or $A^5_{3,3}$ of C_3" (§ 2).

From (23) the identical vanishing of the form

$$(103) \quad (a_3 x)^3 (a_3' x)^3 (a_3'' x)^3 (\alpha_1 \alpha_2)^2 (\alpha_1 y) (\alpha_2 y) \equiv c_0 y_2^2 - 2 c_1 y_1 y_2 + c_2 y_1^2 ,$$

where the symbols α are defined in (102), is the condition that $H_{2,3}$ be an $I_{2,3}$. This gives three relations, $c_k = 0$, each of the ninth degree in x, which must be identically satisfied. If C_3 has an $A^5_{3,3}$ on N_3 with parameters $x_1, \dots, x_5$, the polar quadric of any one has the other four points as an $A^4_{2,2}$, whence it is apolar to all the quadrics containing N_3 and for this particular x the form (103) vanishes independently of y. Hence the quintic $\Pi (x x_i)$, $i = 1, 2, \dots, 5$, is contained as a factor in the three equations, $c_k = 0$. If C_3 has a second $A^5_{3,3}$ on N_3 with points entirely distinct from those of the first, the equations, $c_k = 0$, of degree 9 must be satisfied by the parameters $x_1, \dots, x_{10}$ of the two $A^5_{3,3}$'s and must vanish identically. Thus $H_{2,3}$ is an $I_{2,3}$ if it has two distinct involutive sets. Again let x_1, x_2 be a neutral pair of $H_{2,3}$. Then the polar quadric of x_1 as to C_3 has a double point at x_2, and vice versa; *i. e.*, the point pair x_1, x_2 on N_3 is a pair of corresponding points on the Hessian, H, of C_3. Each polar quadric is apolar to all the quadrics on N_3, whence each equation, $c_k = 0$, has the factor $(x x_1)(x x_2)$. If then $H_{2,3}$ has five distinct neutral pairs, the equations, $c_k = 0$, must vanish identically and $H_{2,3}$ is an $I_{2,3}$. Since the general $I_{2,3}$ arises from the meets of a plane rational quintic, $R_5^{(2)}$, with lines of the plane, and since the neutral pairs of $I_{2,3}$ correspond to the six double points of $R_5^{(2)}$, the existence of five neutral pairs must imply that of a sixth. Hence:

(104) *If the form $H_{2,3}$ has two distinct involutive sets, or if it has five (and à fortiori, six) neutral pairs, the form is an $I_{2,3}$.*

This algebraic result can be applied to the parametric spread, Γ_3. If x_1, x_2 is a neutral pair of $H_{2,3}$, the planes x_1 and x_2 of N_3 must meet in a line of the surface Γ_3. Two neutral pairs of $H_{2,3}$ determine two lines of Γ_3, necessarily skew. Denote by L_i and M_i, $i = 1, 2, \ldots, 6$, the two sets of six lines of a double six of Γ_3. Then, if $H_{2,3}$ is an $I_{2,3}$, N_3 has the six skew lines L_i of a double six of Γ_3 for axes. Using the dual of the theorem that on a cubic surface there is a system of ∞^2 cubic curves which cut a set of lines L_i twice, we find, corresponding to the 72 sets of L_i on Γ_3, 72 systems of ∞^2 curves N_3 with a set of lines L_i as axes. On each N_3 there is a system of ∞^2 five-planes inscribed in Γ_3 and in an $I_{2,3}$ on N_3. These curves N_3 will be called "parametric involution curves of Γ_3."

We shall now seek to determine the $I_{2,3}$ on the involution curve N_3. Keeping in mind the connection of $I_{2,3}$ with $R_5^{(2)}$, we shall call the parameters of the double points d_i of $R_5^{(2)}$, x_{i1}, x_{i2} respectively, $i = 1, 2, \ldots, 6$; and call the parameter of the further meet of the line $\overline{d_i d_k}$ with $R_5^{(2)}$, d_{ik}. The same notation is transferred to the parameters of the planes of N_3. If a line cuts $R_5^{(2)}$ in five points, the corresponding 5-plane is inscribed in Γ_3. The inscribed 5-planes of Γ_3 on N_3 which contain the neutral pair x_{i1}, x_{i2} each contain three other planes whose parameters are determined by a pencil of binary cubics, whence the points X of Γ_3 lie on a line. Since every other neutral pair is contained in some one cubic, this line cuts every line L_k where $k \neq i$, and must be M_i. Thus the three planes of N_3 on any point of M_i together with the two planes of N_3 on L_i make up an inscribed 5-plane of Γ_3. Let $\overline{M_i L_k}$, $i \neq k$, be the meet of the lines M_i and L_k. The planes on $\overline{M_i L_k}$ are x_{k1}, x_{k2}, d_{ik} and on $\overline{M_k L_i}$ are x_{i1}, x_{i2}, d_{ik}; *i. e.*, the plane d_{ik} of N_3 contains the line $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$.

The double six $L_i M_i$ is a self-dual configuration and determines, in addition to the point spread on it, a cubic envelope Δ_3. On Δ_3, of course, the lines are double lines. The other fifteen double lines are $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$. All the planes of N_3 are planes of Δ_3; two planes are on each line L_i, one (namely d_{ik}) on each line $\overline{\overline{M_i L_k}\,\overline{M_k L_i}}$ and none on each line M_i. If therefore the planes of Δ_3 be mapped on the points of a plane Π in the well-known way — by coordinating the six lines L_i to the six fundamental points $\bar{d}_i$ of Π — the planes of N_3 are mapped on a rational quintic $\bar{R}_5^{(2)}$. This $\bar{R}_5^{(2)}$ can be only a projective modification of $R_5^{(2)}$. For the 15 lines $\overline{\bar{d}_i \bar{d}_k}$ cut $\bar{R}_5^{(2)}$ in points with parameters x_{i1}, x_{i2}, x_{k1}, x_{k2}, d_{ik} which are precisely the points of $R_5^{(2)}$ cut out by the line $d_i d_k$. Hence the $I_{2,3}$ of $\bar{R}_5^{(2)}$ is the same as that of $R_5^{(2)}$. Since the $I_{2,3}$ of $\bar{R}_5^{(2)}$ is cut out by the lines of Π, it will be cut out on N_3 by the ∞^2 cubic curves on Δ_3 which have the six lines M_i as axes and which have, therefore, five planes in common with N_3. Summarizing the above, we find that:

(105) *If two distinct* 5-*planes of* N_3 *are inscribed in* Γ_3, *or if* N_3 *has five* (*and à fortiori, six*) *lines* L_i *of* Γ_3 *as axes, then there are* ∞^2 5-*planes of* N_3 *inscribed in* Γ_3 *and lying in an* $I_{2,3}$. *There are* 72 *systems of* ∞^2 *involution curves* N_3. *The* $I_{2,3}$ *on* N_3 *is cut out by the system of involution curves whose axes are the six lines* M_i *which form with* L_i *a double six of* Γ_3. *One finds thus, among the* ∞^5 *inscribed five-planes of* Γ_3, 36 *systems of* ∞^4, *each system being determined by a double six of* Γ_3.

As a corollary there follows:

(106) *The rational invariant system of an* $R_5^{(2)}$ *coincides with the simultaneous rational invariant system of a cubic surface* (*after the adjunction of the irrationality which separates one of the sets of six lines of a double six on the surface*) *and a cubic curve with the six lines as axes.*

This corollary differs in geometric statement, and not in essential algebraic significance, from that derived at once by mapping the plane of $R_5^{(2)}$ on a cubic surface by means of the six double points of $R_5^{(2)}$.

§ 13. *The Apolarity Cubic Surface of the Form* $I_{2,3}$.

We have seen that if C_3 is the apolarity surface of an $I_{2,3}$, then the norm-curve N_3 can be defined to be a proper cubic space curve which cuts the Hessian, H, of C_3 in six pairs of corresponding points. Such cubic curves will be called "apolarity involution curves" of C_3. $I_{2,3}$ contains six absolute constants, while C_3 contains four at most. If, for the general $I_{2,3}$, C_3 itself is general, it must have ∞^2 involution curves. But if C_3 is necessarily subject to k conditions, it must have ∞^{2+k} involution curves. We shall prove that C_3 is general by actually finding involution curves of the general surface.

Let the Sylvester pentahedron of C_3 be $x_1, x_2, \ldots, x_5$, where $\Sigma x_i = 0$. Denote the five planes x_i by Π_i, their ten lines of intersection by Π_{ik}, and their ten points of intersection by Π_{ikl}; $i, k, l = 1, 2, \ldots, 5$. Take for the equations of C_3 and H

$$C_3 = \frac{x_1^3}{\alpha_1^2} + \frac{x_2^3}{\alpha_2^2} + \frac{x_3^3}{\alpha_3^2} + \frac{x_4^3}{\alpha_4^2} + \frac{x_5^3}{\alpha_5^2} = 0,$$

$$H = \frac{\alpha_1^2}{x_1} + \frac{\alpha_2^2}{x_2} + \frac{\alpha_3^2}{x_3} + \frac{\alpha_4^2}{x_4} + \frac{\alpha_5^2}{x_5} = 0.$$

The lines Π_{ik} are on H and the points Π_{ikl} are double points of H. The polar quadrics as to C_3 of points on Π_{ik} are cones with double points at Π_{lmn}; in particular the polar quadric of Π_{ikl} is the pair of planes $\frac{x_m}{\alpha_m} = \frac{x_n}{\alpha_n}$, $\frac{x_m}{\alpha_m} = -\frac{x_n}{\alpha_n}$, which will be denoted respectively by $\begin{pmatrix} m & n \\ + & + \end{pmatrix}$ and $\begin{pmatrix} m & n \\ + & - \end{pmatrix}$. The pencil of polar quadric cones of points on Π_{ik} have four common generators,

$$\pm \frac{x_l}{\alpha_l} = \pm \frac{x_m}{\alpha_m} = \pm \frac{x_n}{\alpha_n}.$$

These lines will be denoted by $\begin{pmatrix} l & m & n \\ \pm & \pm & \pm \end{pmatrix}$.

Thus the plane $\begin{pmatrix} i & k \\ + & + \end{pmatrix}$ contains the six lines $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$, $\begin{pmatrix} i & k & m \\ + & + & \pm \end{pmatrix}$ and $\begin{pmatrix} i & k & n \\ + & + & \pm \end{pmatrix}$. The lines of a pair such as $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$ meet in a point Π_{ikl}, while the lines of the pair $\begin{pmatrix} i & k & l \\ + & + & \pm \end{pmatrix}$ meet those of the pair $\begin{pmatrix} i & k & m \\ + & + & \pm \end{pmatrix}$ in the four points $\begin{pmatrix} i & k & l & m \\ + & + & \pm & \pm \end{pmatrix}$, where for example $\begin{pmatrix} 1 & 2 & 3 & 4 \\ + & - & + & - \end{pmatrix}$ denotes the point with coordinates $\alpha_1, -\alpha_2, \alpha_3, -\alpha_4, -(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)$. Evidently there are 40 points $\begin{pmatrix} i & k & l & m \\ \pm & \pm & \pm & \pm \end{pmatrix}$, the 8 just written being the poles of the plane Π_n of the Sylvester pentahedron.

To obtain another property of the 40 lines $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ quite useful for our purpose let y be a point on H, whence $\frac{\alpha^2}{y}$ is its corresponding point and $\Sigma y_i = 0$, $\Sigma \frac{\alpha_i^2}{y_i} = 0$. The points y and $\frac{\alpha^2}{y}$ will lie on a line with Π_{ikl} if $\lambda y_j + \frac{\alpha_j^2}{y_j} = 0$, $j = i, k, l$. Hence y satisfies the relations $\frac{y_i^2}{\alpha_i^2} = \frac{y_k^2}{\alpha_k^2} = \frac{y_l^2}{\alpha_l^2}$; *i. e.*, it lies on one of the four lines $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$.

Collecting these facts, we have:

(107) *There is a set of* 40 *lines each of which passes through a double point of* H *and meets* H *further in a pair of corresponding points. The four lines* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ *meet in the point* Π_{ikl}. *Cones on these four lines are the polar quadrics of points on the line* Π_{mn}. *The lines* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$ *and* $\begin{pmatrix} l & m & n \\ \pm & \pm & \pm \end{pmatrix}$ *are skew; the lines* $\begin{pmatrix} i & k & l \\ \pm & + & + \end{pmatrix}$ *and* $\begin{pmatrix} k & l & m \\ + & - & \pm \end{pmatrix}$ *are skew; while the lines* $\begin{pmatrix} i & k & l \\ \pm & + & + \end{pmatrix}$ *and* $\begin{pmatrix} k & l & m \\ + & + & \pm \end{pmatrix}$ *are incident at the point* $\begin{pmatrix} i & k & l & m \\ \pm & + & + & \pm \end{pmatrix}$ *which is a pole of* Π_n *and in the plane* $\begin{pmatrix} k & l \\ + & + \end{pmatrix}$ *which is a part of the polar quadric of* Π_{imn}.

Two other groupings of points will be used. The eight poles of the plane Π_n, namely $\begin{pmatrix} i & k & l & m \\ \pm & \pm & \pm & \pm \end{pmatrix}$, divide into two sets of four denoted by t_{n+} and t_{n-} according as the number of negative signs is even or odd. It is evident from the coordinates of these points that:

(108) *The eight poles of* Π_n *divide into two tetrahedra* t_{n+} *and* t_{n-} *which with the tetrahedron* $\tau_n = \Pi_i \Pi_k \Pi_l \Pi_m$ *form a desmic system. The* 16 *lines of perspection of the system are* $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$.

We consider now the five points $\begin{pmatrix} i & k & l & m \\ + & + & + & + \end{pmatrix}$, which we denote collectively by $(+ + + + +)$ and singly by $p_1, p_2, \ldots, p_5$. The five points are poles of the five planes Π_i. The ten lines which join them are on the ten points Π_{ik}. The ten planes which join them are the planes $\begin{pmatrix} i & k \\ + & + \end{pmatrix}$. The mixed polar planes of the points two at a time are the planes $\begin{pmatrix} i & k \\ + & - \end{pmatrix}$. We find in all 16 such sets of five points corresponding to the 16 symbols $(\pm \pm \pm \pm \pm)$. A fairly inclusive statement of the incidence relations in our configuration is this:

(109) *The* 40 *poles of the five Sylvester planes can be divided in* 16 *ways into sets of five points such that each set of five points and the ten planes on them, together with the Sylvester five planes and the ten points on them, form a Desargues configuration* $(15_6, 20_3)$. *In each configuration ten lines are determined by the*

Sylvester planes; the other ten lines are found among the 40 $\begin{pmatrix} i & k & l \\ \pm & \pm & \pm \end{pmatrix}$. *Each of the sixteen configurations is self-polar as to a quadric of the type*

$$\frac{x_1^2}{\pm\alpha_1}+\frac{x_2^2}{\pm\alpha_2}+\cdots\cdot+\frac{x_5^2}{\pm\alpha_5}.$$

Writing C_3 symbolically as $a_x^3=b_x^3=\ldots.$, we have the plane equation of the polar quadric of a point x, $a_x b_x c_x (abcu)^2=0$. If x and y are corresponding points on H, then

$$a_x b_x c_x (abcu)^2=\rho\,.\,u_y^2$$

and

$$a_y b_y c_y (abcu)^2=\sigma\,.\,u_x^2\,.$$

Let $\alpha=\alpha_x^2$ be any quadric; it determines an "associated cubic surface," $C_\alpha=a_x b_x c_x (abc\alpha)^2$, the locus of points x whose polar quadrics are apolar to α. C_α contains the ten points Π_{ikl}, and as α changes C_α runs through the linear system of cubics on these ten points. C_α meets H in a 12-ic curve, the locus of points which correspond to the points of the octavic curve in which α meets H. Let x be a point on H, α, and C_α, and let y be its corresponding point. Then y lies on H, C_α, and α; *i. e.*, the sextic curve in which α and C_α meet cuts H in twelve pairs of corresponding points.

(110) *There is a system of* ∞^9 *sextic curves, the meets of quadrics* α *with their associated cubics* C_α *which cut the Hessian of a cubic surface in twelve pairs of corresponding points.*

If α contains a point Π_{ikl}, C_α contains the line Π_{mn}. If α contains the four points τ_i, C_α contains the four lines Π_{ik}. Hence the plane Π_i which meets C_α in these four lines must be a part of C_α. The other part of C_α must be a quadric which also is on the vertices τ_i. Thus each quadric on τ_i determines another of this same kind. Let us consider this correspondence for the case $i=5$. A quadric on τ_5 is $\Sigma\, b_{ik} x_i x_k=0$, where $i, k=1, 2, 3, 4$ and $i\neq k$. The polar quadric of x in planes is $\sum\limits_{l,m,n=1}^{5}\frac{x_l x_m x_n}{\alpha_l^2\alpha_m^2\alpha_n^2}\Pi_{lmn}^2$. This is apolar to $\Sigma\, b_{ik} x_i x_k$ if $\sum\limits_{i,k=1}^{4}\frac{b_{ik}}{\alpha_l^2\alpha_m^2\alpha_n^2}x_l x_m x_n=0$; *i. e.*, if $\frac{x_5}{\alpha_5^2}\sum\limits_{i,k=1}^{4} b_{ik}\alpha_i^2\alpha_k^2 x_l x_m=0$. Hence the relation between $\Sigma\, b_{ik} x_i x_k=0$ and its associated quadrics $\Sigma\, b'_{ik} x_i x_k=0$ is given by

$$b'_{12}=\alpha_3^2\alpha_4^2 b_{34},\quad b'_{13}=\alpha_2^2\alpha_4^2 b_{24},\quad b'_{14}=\alpha_2^2\alpha_3^2 b_{23},$$
$$b'_{34}=\alpha_1^2\alpha_2^2 b_{12},\quad b'_{24}=\alpha_1^2\alpha_3^2 b_{13},\quad b'_{23}=\alpha_1^2\alpha_4^2 b_{14}.$$

Evidently this relation is involutory and the two nets of quadrics

$$\lambda\left(\frac{x_1 x_2}{\alpha_1\alpha_2}-\frac{x_3 x_4}{\alpha_3\alpha_4}\right)+\mu\left(\frac{x_1 x_3}{\alpha_1\alpha_3}-\frac{x_4 x_2}{\alpha_4\alpha_2}\right)+\nu\left(\frac{x_1 x_4}{\alpha_1\alpha_4}-\frac{x_2 x_3}{\alpha_2\alpha_3}\right)=0,$$

$$\rho\left(\frac{x_1 x_2}{\alpha_1\alpha_2}+\frac{x_3 x_4}{\alpha_3\alpha_4}\right)+\sigma\left(\frac{x_1 x_3}{\alpha_1\alpha_3}+\frac{x_4 x_2}{\alpha_4\alpha_2}\right)+\tau\left(\frac{x_1 x_4}{\alpha_1\alpha_4}+\frac{x_2 x_3}{\alpha_2\alpha_3}\right)=0$$

are each self-associated. Denote these nets by Q_{5+} and Q_{5-} respectively. The net Q_{5+} is on the eight points τ_5 and t_{5+}; the net Q_{5-} is on the eight points τ_5 and t_{5-}. On account of the involutory relation each quadric of the one net meets each quadric of the other in a quartic curve which is on the four points τ_5 and which meets H further in four pairs of corresponding points. Since each quadric in a net is associated to itself, it meets H in an octavic curve which has double points at the vertices of τ_5 and which contains ∞^1 pairs of corresponding points. Hence:

(111) *The ten nets of quadrics $Q_{i\pm}$ on the eight points of τ_i and $t_{i\pm}$ meet H in ten systems of ∞^2 self-corresponding octavic curves. The meets of the nets Q_{i+} and Q_{i-} determine five systems of ∞^4 quartic curves on the vertices of τ_i which meet H again in four pairs of corresponding points. The meets of the nets $Q_{i\pm}$ and $Q_{k\pm}$ determine forty systems of ∞^4 quartic curves which pass through Π_{lmn} and meet H again in seven pairs of corresponding points.*

To locate involution cubic curves in one of the forty systems just mentioned let us take, to fix ideas, the nets Q_{4-} and Q_{5-}. Their quartic curves of intersection pass through Π_{123}. Since, as we shall see, an involution cubic curve can not in general pass through a double point of H, if one such quartic curve is to degenerate into a line and a cubic curve, the line must pass through Π_{123} and meet H again in a pair of corresponding points. It must therefore be one of the four lines $\begin{pmatrix}1&2&3\\ \pm&\pm&\pm\end{pmatrix}$, and we shall take it to be $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$. The net Q_{4-} is on the base-points τ_4 and t_{4-}. Since these are desmic, the cones of the net are found in four pencils with double points at t_{4+}. Of the vertices of t_{4+}, a single one, namely $p_4=\alpha_1,\ \alpha_2,\ \alpha_3,\ \backslash,\ \alpha_5$,[*] is found on the chosen line $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$. The pencil of cones with vertex at p_4 and on τ_4 and t_{4-} has the base-generators $\begin{pmatrix}1&2&3\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}2&3&5\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}1&3&5\\ +&+&+\end{pmatrix}$, $\begin{pmatrix}1&2&5\\ +&+&+\end{pmatrix}$. Similarly, in the net Q_{5-}

---

[*] Here and elsewhere one of the five coordinates, the negative of the sum of the other four, is indicated by the stroke \.

there is a single pencil of cones with vertex at a point p_5 on $\begin{pmatrix}1 & 2 & 3\\ + & + & +\end{pmatrix}$, and this has the base-generators $\begin{pmatrix}1 & 2 & 3\\ + & + & +\end{pmatrix}$, $\begin{pmatrix}2 & 3 & 5\\ + & + & +\end{pmatrix}$, $\begin{pmatrix}1 & 3 & 5\\ + & + & +\end{pmatrix}$, $\begin{pmatrix}1 & 2 & 5\\ + & + & +\end{pmatrix}$. According to (107) the common generator of the two pencils is met by each of the other three base-generators and each one of these three meets a definite one of the three in the other pencil in a point p_i, $i = 1, 2, 3$. Hence every cone of the one pencil cuts every cone of the other in the line $\begin{pmatrix}1 & 2 & 3\\ + & + & +\end{pmatrix}$ and a cubic curve on the five points p_1, p_2, p_3, p_4, p_5. Moreover every such cubic curve lies on a cone of each pencil. Making use of (109) we obtain the desired result:

(112) *There are* 16 *systems of* ∞^2 *apolarity involution curves of a general cubic surface* C_3. *Each system is the totality of proper cubic curves on one of the* 16 *sets of five points* $(\pm\ \pm\ \pm\ \pm\ \pm)$.

We still have to show that, for the general $I_{2,3}$, the involution curve N_3 is found in one of the above 16 systems. Note first that N_3 can not meet a line Π_{ik}. For if it meets Π_{ik} at a point, it must contain the correspondent of the point, Π_{lmn}; in which case it could not meet H in six distinct pairs of corresponding points. Denote by C_{ikl} the chord of N_3 from the point Π_{ikl}.

Since the polar quadric of any point on N_3 is apolar to all the point quadrics on N_3, the cubic surface C_α contains N_3 if the quadric α contains N_3. Let β be a quadric on N_3. Then β and C_β meet in a sextic curve made up of N_3 and another cubic curve N_β which cuts N_3 in five points and H in six pairs of corresponding points. Hence N_β is also an involution cubic and we find from the net β, ∞^2 involution curves N_β which will be called "incident" curves of N_3.

Let β_5 be the particular quadric on N_3 and the two points Π_{123} and Π_{124}. Then C_{β_5} contains N_3 and the lines Π_{45} and Π_{35}. The plane Π_5 meets C_{β_5} in these two lines and a further line which must contain the point Π_{125} and the three points where N_3 meets Π_5. Since this is impossible, Π_5 is part of C_{β_5} and β_5 contains all four points τ_5. C_{β_5} contains N_3 and the same four points, whence C_{β_5} is $\Pi_5 . \beta_5$, or β_5 is a self-associated quadric (see p. 341), which we shall suppose to be in the net Q_{5-}. Similarly, there is a self-associated quadric β_4 on N_3 and the points τ_4, which we shall suppose to be in the net Q_{4-}.

The pencil of quadrics $\beta_4 + \lambda\,\beta_5$ contains N_3 and C_{123}; the associated pencil $C_{\beta_4} + \lambda\, C_{\beta_5}$ contains N_3, C_{123}, and Π_{45}; it is in fact $\frac{y_4}{\alpha_4}\beta_4 + \lambda\frac{y_5}{\alpha_5}\beta_5$. Each quadric of the pencil meets its associate in N_3, C_{123}, and a conic. The locus of these conics is evidently the plane $\frac{y_4}{\alpha_4} - \frac{y_5}{\alpha_5} = 0$ or $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$. The conics form a pencil on the points q_1, q_2, q_3, where $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$ meets N_3, and the point q where C_{123} meets $\begin{pmatrix} 4 & 5 \\ + & + \end{pmatrix}$. But each conic of the pencil together with C_{123} is a degenerate "incident" involution curve N_β and must cut H in six pairs of corresponding points. From the symmetry of the case it is necessary that these pairs be distributed as follows: three on the conic, two for the meets of the conic with Π_{45} and the point Π_{123} on C_{123}, and therefore one further pair on C_{123}. This requires however that C_{123} be one of the four lines $\begin{pmatrix} 1 & 2 & 3 \\ \pm & \pm & \pm \end{pmatrix}$; let it be $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$. But one verifies at once that if β_5 is a quadric in the net Q_{5-} and contains the line $\begin{pmatrix} 1 & 2 & 3 \\ + & + & + \end{pmatrix}$, it must be a cone with vertex at p_5. Similarly, β_4 is a cone of the net Q_{4-} with vertex at p_4. These two cones meet in the given N_3 and they are also precisely the cones used to determine one of the 16 systems of involution curves already found; *i. e.*, the general involution curve is found in one such system. The two possible choices of β_5 in a net $Q_{5\pm}$ and β_4 in a net $Q_{4\pm}$ and the four possible choices of the line $\begin{pmatrix} 1 & 2 & 3 \\ \pm & \pm & \pm \end{pmatrix}$ give rise to the 16 systems.

(113) Incidentally we have shown that:

In addition to the ten singular planes of the Cayleyan congruence which contain a point Π_{ikl} *and line* Π_{lm}, *the lines of the congruence being on the point, there is a set of twenty singular planes* $\begin{pmatrix} i & k \\ \pm & \pm \end{pmatrix}$. *Such a plane cuts the Hessian in the line* Π_{ik} *and a cubic curve which contains* ∞^1 *pairs of corresponding points whose joins envelope a general cubic.*

We remarked earlier that if the apolarity and parametric spreads are of the same dimensions they are covariants of each other and *a common involution norm-curve.* In our present case the connection between Γ_3 and C_3 can be readily exhibited by means of the null-system or correlation set up by N_3. The planes x_1, x_2, x_3 of N_3 which meet in a point X of Γ_3 are transformed by this correlation into points x_1, x_2, x_3 on N_3 which form an apolar triad of C_3. Hence Γ_3 is transformed into a cubic envelope E_3 whose planes cut N_3 in apolar triads of C_3. The six lines L_i of Γ_3 become the six lines L_i', joining the pairs of corresponding points in which N_3 meets H. Hence these six lines of the Cayleyan congruence belong to a double six. The six lines M_i of Γ_3 become the cross-lines, M_i', of L_i'. The cubic envelope Δ_3 on the double six $L_i M_i$ becomes a cubic surface D_3 which has the double six $L_i' M_i'$. Making use of (105), we have the theorem:

(114) *An involution curve N_3 of C_3 cuts H in six pairs of corresponding points. The six lines of the Cayleyan congruence so determined are skew lines L_i' of a double six. The cubic surface D_3 on N_3, on the six lines L_i', and on their cross-lines M_i', contains ∞^2 cubic curves with the chords M_i'. These curves cut out on N_3 the given $I_{2,3}$.*

§ 14. *The Apolarity Spread of the Rational Plane Curve of Order n.*

Some of the foregoing results concerning the apolarity cubic curve in S_2 of an $R_4^{(2)}$ and concerning the apolarity cubic surface in S_3 of an $R_5^{(2)}$ can be readily generalized. The $R_n^{(2)}$ establishes its $I_{2,n-2}$, the involution cut out by the lines of the plane. This we express by means of a symmetric form,

$$H_{2,n-2} = (a_1 x_1)^{n-2} (a_2 x_2)^{n-2} (a_3 x_3)^{n-2}.$$

The apolarity spread of the form is a cubic spread, $C_3^{(n-2)}$, in S_{n-2} referred to a norm-curve, N_{n-2}. The condition that $H_{2,n-2}$ be an $I_{2,n-2}$ is the identical vanishing of a form, F, of order $(n-2)^2$ in x and symmetric in x_1 and x_2 of order $(n-4)$ in each. The argument of § 12 for the case $n=5$ can be extended. If $H_{2,n-2}$ has an involutive set, $y_1, y_2, \ldots, y_n$, then F contains the factor, $(xy_1) \ldots (xy_n)$; or if $H_{2,n-2}$ has a neutral pair, y_1, y_2, then F contains the factor $(xy_1)(xy_2)$. If the number of these distinct factors be greater than $(n-2)^2$, F must vanish identically.

(115) *The form $H_{2,n-2}$ is an $I_{2,n-2}$ if it has an integer number greater than $\frac{(n-2)^2}{n}$ of distinct involutive sets; or if it has an integer number greater than $\frac{(n-2)^2}{2}$ $\left(\text{and à fortiori equal to } \frac{(n-1)(n-2)}{2}\right)$ of distinct neutral pairs.*

A geometric statement of the theorem is this:

(116) *If a norm-curve in S_{n-3} contains more than $\frac{(n-2)^2}{n}$ distinct antorthic sets $A_{3,3}^n$ of a cubic spread $C_3^{(n-2)}$, or if it cuts the Hessian of $C_3^{(n-2)}$ in more than $\frac{(n-2)^2}{2}$ pairs of corresponding points, then it contains ∞^2 $A_{3,3}^n$'s of $C_3^{(n-2)}$ which lie in an $I_{2,n-2}$.*

We are concerned mainly with the case of an $I_{2,n-2}$ and the $R_n^{(2)}$ from which it originates. $C_3^{(n-2)}$ cuts N_{n-2} in the $3(n-2)$ points whose parameters give the flexes of $R_n^{(2)}$. Caporali has shown for the cases $n=4, 5$ that if $n-1$ points of an $A_{3,3}^n$ of $C_3^{(n-2)}$ lie on $C_3^{(n-2)}$, the n-th point also lies on $C_3^{(n-2)}$. His method of proof applies equally well to the general case and we translate the result thus:

(117) *If $(n-1)$ flexes of a rational plane curve of order n lie on a line, an n-th flex also lies on the line.*

We have seen that $C_3^{(2)}$ and $C_3^{(3)}$ were general cubic spreads. $C_3^{(n-2)}$ has $\frac{1}{6}(n^3-6n^2+11n-6)$ absolute constants, while the $R_n^{(2)}$ has $3(n-3)$ absolute constants. Certainly, then, when $n > 6$, the apolarity spread of an $I_{2,n-2}$ is a *special* cubic spread. It would be interesting to know more of the conditions on the spread and of the location of the involution curves than is indicated in (116). These questions are answered in this paragraph for only two kinds of rational curves.

Take first the $R_n^{(2)}$ with an $(n-1)$-fold point. The $I_{2,n-2}$ has a set of parameters, $\lambda_1, \ldots, \lambda_{n-1}$, such that any two of the set are a neutral pair. There is on N_{n-2} a set of $n-1$ points such that any two are apolar to the cubic $C_3^{(n-2)}$. Using these as an $(n-1)$-point of reference, $C_3^{(n-2)}$ becomes

$$a_1 x_1^3 + \cdots + a_{n-1} x_{n-1}^3,$$

whence:

(118) *If $C_3^{(n-2)}$ can be expressed as a sum of $(n-1)$ cubes, any one of the $\infty^{2(n-3)}$ proper norm-curves N_{n-2} on the points of its orthic $(n-1)$-edron is an involution curve. The $A_{3,3}^n$'s of C_3^{n-2} on such a curve lie in the $I_{2,n-2}$ of an $R_n^{(2)}$ with an $(n-1)$-fold point.*

To obtain the second kind of $R_n^{(2)}$ considered here we assume that $C_3^{(n-2)}$ has an orthic n-edron, $\Pi_1, \Pi_2, \ldots, \Pi_n$, and then prove that it has involution curves. Using the n-edron as a basis of reference, we have

$$C_3^{(n-2)} = \sum_{i=1}^{n} \frac{x_i^3}{\alpha_i^2}, \text{ where } \sum_{i=1}^{n} x_i = 0.$$

The Hessian of order $n-1$ is

$$H = \sum_{1}^{n} \frac{\alpha_i^2}{x}.$$

If both $\Sigma y_i = 0$ and $\Sigma \frac{\alpha_i^2}{y_i} = 0$, the points y_i and $\frac{\alpha_i^2}{y_i}$ are corresponding points on H. H has simple points on the spaces Π_{ik}, double points on the spaces

Π_{ikl}, etc., the notation here being similar to that of the preceding paragraph. The configuration of poles of the spaces Π_i obtained there exists for all values of n greater than two. In particular from the n sets of 2^{n-2} poles one can in 2^{n-1} ways pick out n poles like the following:

$$\begin{array}{llllll}
1/\alpha_1\, p_1 = \alpha_1 - \alpha, & \alpha_2, & \alpha_3, & \ldots\ldots, & \alpha_n, \\
1/\alpha_2\, p_2 = \alpha_1, & \alpha_2 - \alpha, & \alpha_3, & \ldots\ldots, & \alpha_n, \\
1/\alpha_3\, p_3 = \alpha_1, & \alpha_2, & \alpha_3 - \alpha, & \ldots\ldots, & \alpha_n, \\
\multicolumn{5}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,} \\
1/\alpha_n\, p_n = \alpha_1, & \alpha_2, & \alpha_3, & \ldots\ldots, & \alpha_n - \alpha,
\end{array}$$

where $\alpha = \sum\limits_i \alpha_i$ and $\sum\limits_i p_i \equiv 0$.

The point p_i is a pole of Π_i and the line $\overline{p_i p_k}$ passes through the $(n-3)$-fold point $\Pi_{l,m,n,\ldots\ldots}$ of the Hessian.

A norm-curve on the n points p_i and the point y is unique. Its parametric equation is:

$$\begin{array}{lrl}
(119) & 1/\alpha_1\, x_1 = & \phantom{\alpha_1 \overline{21}\, t_{21} +{}} \alpha_2\, \overline{12}\, t_{12} + \alpha_3\, \overline{13}\, t_{13} + \ldots\ldots + \alpha_n\, \overline{1n}\, t_{1n}, \\
& 1/\alpha_2\, x_2 = & \alpha_1\, \overline{21}\, t_{21} \phantom{{}+ \alpha_2 \overline{12}\, t_{12}} + \alpha_3\, \overline{23}\, t_{23} + \ldots\ldots + \alpha_n\, \overline{2n}\, t_{2n}, \\
& 1/\alpha_3\, x_3 = & \alpha_1\, \overline{31}\, t_{31} + \alpha_2\, \overline{32}\, t_{32} \phantom{{}+ \alpha_3 \overline{13}\, t_{13}} + \ldots\ldots + \alpha_n\, \overline{3n}\, t_{3n}, \\
& \multicolumn{2}{l}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,} \\
& 1/\alpha_n\, x_n = & \alpha_1\, \overline{n1}\, t_{n1} + \alpha_2\, \overline{n2}\, t_{n2} + \alpha_3\, \overline{n3}\, t_{n3} + \ldots\ldots,
\end{array}$$

where $\overline{ik} = \dfrac{y_i}{\alpha_i} - \dfrac{y_k}{\alpha_k}$ and $t_{ik} = \dfrac{\prod\limits_1^n \left(t - \dfrac{y_l}{\alpha_l}\right)}{\left(t - \dfrac{y_i}{\alpha_i}\right)\left(t - \dfrac{y_k}{\alpha_k}\right)}$.

The parameter of p_i is $\dfrac{y_i}{\alpha_i}$; the parameter of y is ∞. When $t = 0$, we have the point

$$1/\alpha_i\, x_i^{(0)} = \left[\left(\frac{\alpha_1^2}{y_1} + \ldots\ldots\right) - \left(\frac{\alpha_1 \alpha_i}{y_i} + \ldots\ldots\right)\right] \prod_1^n \left(\frac{y_k}{\alpha_k}\right),$$

whence

$$x_i^{(0)} = \left[\alpha_i \left(\sum_1^n \frac{\alpha_k^2}{y_k}\right) - \frac{\alpha_i^2}{y_i}\left(\sum_1^n \alpha_k\right)\right] \prod_1^n \left(\frac{y_k}{\alpha_k}\right).$$

Since y can be any point on N_{n-2}, let it be on H, whence

$$x_i^{(0)} = -\left(\sum_1^n \alpha_k\right) \prod_1^n \left(\frac{y_k}{\alpha_k}\right) \cdot \frac{\alpha_i^2}{y_i};$$

i. e., $x_i^{(0)}$ is the corresponding point of y. Hence N_{n-2} cuts H in pairs of corresponding points and is an involution curve.

(120) *If a cubic spread $C_3^{(n-2)}$ in S_{n-2} can be expressed as a sum of n cubes, one can pick from the $2^{n-2}\,.\,n$ poles of its n orthic S_{n-3}'s in 2^{n-1} ways an n-point with the following property: the n points and the $\frac{1}{2}\,n\,(n-1)$ S_{n-3}'s on them together with the n orthic S_{n-3}'s and the $\frac{1}{2}\,n\,(n-1)$ points on them form a Desargues configuration (a section of the complete $(n+1)$-point in S_{n-1}). Every proper norm-curve N_{n-2} on such an n-point cuts the Hessian of $C_3^{(n-2)}$ in $\frac{1}{2}\,(n-1)\,(n-2)$ pairs of corresponding points and contains $\infty^2\,A_{3,}^n$'s of $C_3^{(n-2)}$ which lie in the $I_{2,\,n-2}$ of a rational plane curve $R_n^{(2)}$ which is subject to $(n-5)$ conditions.*

For $C_3^{(n-2)}$ has only $n-1$ absolute constants and only $n-3$ are added in the choice of an N_{n-2}.

With each of the 2^{n-1} systems of involution curves there is associated a quadric with regard to which the corresponding Desargues configuration is self-polar. For the particular n-point, p_i, this quadric is

$$Q \equiv \frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} + \ldots. + \frac{x_n^2}{\alpha_n}.$$

Let Q_x be the polar quadric of a point x as to $C_3^{(n-2)}$. Then

$$Q_{p_1} = \frac{\alpha_1 - \alpha}{\alpha_1}\,\frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} + \ldots. + \frac{x_n^2}{\alpha_n};$$

i. e., Q and Q_{p_1} touch along the space Π_1, which space has the same pole, p_1, as to both quadrics. Thus p_1 and any self-polar $(n-2)$-point of Q or Q_{p_1} in Π_1 is a self-polar $(n-1)$-point of Q and Q_{p_1}. But it is known that all the ∞^{n-3} norm-curves in our system cut Π_1 in self-polar $(n-2)$-points of Q. Thus p_1 and the $n-2$ points in which Π_1 cuts N_{n-2} constitute a self-polar $(n-1)$-point of Q on N_{n-2}. The existence of one such set implies that of ∞^1 sets which lie in an $I_{1,\,n-2}$, and Q is apolar to all the point quadrics on N_{n-2}. In this $I_{1,\,n-2}$ each of the n sets of points (p_i and the points on Π_i) is also a self-polar $(n-1)$-point of of a polar quadric Q_{p_i}, hence with the point p_i forms an $A_{3,\,3}^n$ of $C_3^{(n-2)}$. A similar fact is true of the polar quadric of any point on N_{n-2}, and we shall verify the theorem:

(121) *If $Q = \Sigma\,\dfrac{x_i^2}{\alpha_i}$ is the quadric associated with the system of ∞^{n-3} involution curves in which the given N_{n-2} lies, then the polar quadric of a point τ on N_{n-2}*

has in common with Q a self-polar $(n-1)$-point whose $(n-1)$ vertices t are on N_{n-2}. The $(1, n-1)$ relation between τ and t is

$$\alpha\left\{\prod_{1}^{n}\left(t-\frac{y_k}{\alpha_k}\right)\right\}_{\tau}-\frac{1}{n}(\tau t)\left\{\frac{x_1}{\alpha_1}+\frac{x_2}{\alpha_2}+\dots\dots+\frac{x_n}{\alpha_n}\right\}=0,$$

where the subscript of the brace indicates the first polar of τ, and where $\frac{x_i}{\alpha_i}$ is the $(n-2)$-ic in t of (119).

The point $t=\tau$ and the $n-1$ points t determined by τ together constitute an $A^n_{3,3}$ of C_3^{n-2}, whence by varying τ in the $(2, n)$ relation

$$(\tau t)\,\alpha\left\{\prod_{1}^{n}\left(t-\frac{y_k}{\alpha_k}\right)\right\}_{\tau}-\frac{1}{n}(\tau t)^2\left\{\Sigma\frac{x_k}{\alpha_k}\right\}=0,$$

a singly infinite quadratic system of $A^n_{3,3}$'s is obtained.

Note first that in (119) $\frac{x_1}{\alpha_1}-\frac{x_i}{\alpha_i}=\alpha\,(1\,i)\,t_{1i}$ and the $(1, n-1)$ relation can be written

$$\alpha\left\{\prod_{1}^{n}\left(t-\frac{y_k}{\alpha_k}\right)\right\}_{\tau}-\frac{1}{n}(\tau t)\left\{n\frac{x_1}{\alpha_1}-\alpha\left[\overline{12}\,t_{12}+\dots\dots+\overline{1n}\,t_{1n}\right]\right\}.$$

Let τ take the value $\frac{y_1}{\alpha_1}$. The polarized term becomes

$$\frac{1}{n}\left(t-\frac{y_1}{\alpha_1}\right)\left[\overline{12}\,t_{12}+\dots\dots+\overline{1n}\,t_{1n}\right],$$

which is cancelled by part of the second term, whence the result is $\left(t-\frac{y_1}{\alpha_1}\right)\frac{x_1}{\alpha_1}$. Similarly for $\tau=\frac{y_i}{\alpha_i}$, the $(n-1)$-ic in t becomes $\left(t-\frac{y_i}{\alpha_i}\right)\frac{x_i}{\alpha_i}$. We have identified these $(n-1)$-ics as self-polar $(n-1)$-points of Q and two alone are sufficient to define the pencil. We have also identified the n-ics $\left(t-\frac{y_i}{\alpha_i}\right)^2\frac{x_i}{\alpha_i}$ as $A^n_{3,3}$'s of $C_3^{(n-2)}$. The ∞^2 $A^n_{3,3}$'s lie in a linear system which must contain the ∞^1 given by varying τ in the $(2, n)$ relation, since they have n known members in common. We shall recur to the meaning of this $(1, n-1)$ relation with regard to the underlying $R_n^{(2)}$ in the next paragraph.

It is evident from the analysis that our entire apparatus is determined by the choice of the n-space Π_i, of the quadric Q for which the n-space is orthic, and of the N_{n-2} on the orthic n-point p_i of Q which is the polar of the given n-space Π_i as to Q. Since N_{n-2} contains one orthic n-point of Q, it is known to contain ∞^3 which lie in an $I_{3,\,n-3}$. These are the ∞^3 canonizant sets of the $2(n-2)$-ic determined on N_{n-2} by the S_{n-3}'s common to Q and N_{n-2}. In this $I_{3,\,n-3}$ all the sets of our given $I_{2,\,n-2}$ are found. To prove this, we have merely to show that if x and y are points on N_{n-2} their polar space as to $C_3^{(n-2)}$ has a pole as to Q which is on the line $\overline{xy}$. The condition that x and y be on N_{n-2} is the vanishing of the matrix

$$\left\Vert\begin{matrix} \dfrac{x_1}{\alpha_1} & \dfrac{x_2}{\alpha_2} & \cdots\cdots & \dfrac{x_n}{\alpha_n} \\ \dfrac{x_1y_1}{\alpha_1^2} & \dfrac{x_2y_2}{\alpha_2^2} & \cdots\cdots & \dfrac{x_ny_n}{\alpha_n^2} \\ \dfrac{y_1}{\alpha_1} & \dfrac{y_2}{\alpha_2} & \cdots\cdots & \dfrac{y_n}{\alpha_n} \\ 1 & 1 & \cdots\cdots & 1 \end{matrix}\right\Vert$$

Each determinant of this matrix is, for given y on N_{n-2}, a quadric on N_{n-2}; *e. g.*, the first determinant is

$$\begin{aligned} &\overline{12}\,\overline{34}\left(\frac{x_3x_4}{\alpha_3\alpha_4}+\frac{x_1x_2}{\alpha_1\alpha_2}\right)+\overline{13}\,\overline{42}\left(\frac{x_4x_2}{\alpha_4\alpha_2}+\frac{x_1x_3}{\alpha_1\alpha_3}\right)+\overline{14}\,\overline{23}\left(\frac{x_2x_3}{\alpha_2\alpha_3}+\frac{x_1x_4}{\alpha_1\alpha_4}\right)\\ &\quad=\overline{12}\,\overline{34}\left(\frac{x_4}{\alpha_4}-\frac{x_2}{\alpha_2}\right)\left(\frac{x_3}{\alpha_3}-\frac{x_1}{\alpha_1}\right)+\overline{13}\,\overline{42}\left(\frac{x_2}{\alpha_2}-\frac{x_1}{\alpha_1}\right)\left(\frac{x_4}{\alpha_4}-\frac{x_3}{\alpha_3}\right)\\ &\quad=\overline{12}\,\overline{13}\,\overline{34}\,\overline{42}\,[t_{12}t_{34}-t_{13}t_{42}]\equiv 0. \end{aligned}$$

But the vanishing of the matrix is also the condition that constants $\varkappa$, λ, μ, ν can be found such that

$$\varkappa\Sigma\frac{x_i}{\alpha_i}z_i+\lambda\Sigma\frac{y_i}{\alpha_i}z_i+\mu\Sigma\frac{x_iy_i}{\alpha_1^2}z_i+\nu\Sigma z_i\equiv 0;$$

i. e., that the polar space of $\varkappa x+\lambda y$ as to Q is the polar space of xy as to $C_3^{(n-2)}$.

The n-space Π_i determines an n-point, one of the points being $-(n-1)$, $1, 1, \ldots, 1$. This n-point is perspective to the n-point p_i. The center of perspection is the point $\alpha_1-\frac{1}{n}\alpha$, $\alpha_2-\frac{1}{n}\alpha$, $\ldots$, $\alpha_n-\frac{1}{n}\alpha$. Dually the n-point p_i determines an n-space perspective to the n-space Π_i and the S_{n-3}

of perspection is $\sum_1^n \frac{x_i}{\alpha_i} = 0$. The complete Desargues configuration determines $n+1$ such S_{n-3}'s. The $C_3^{(n-2)}$ is defined by the requirements that it have the orthic n-space Π_i whose poles as to $C_3^{(n-2)}$ are points p_i.

(122) *Given a quadric* (Q) *apolar to all the point quadrics on a norm-curve* (N_{n-2}) *in* S_{n-2}, *then there exists* ∞^3 *n-points on the norm-curve which are orthic to the quadric and which lie in an* $I_{3,\,n-3}$. *If one of these n-points* (p_i) *be isolated, its polar orthic n-space* (Π_i) *as to the quadric determines with it a unique* S_{n-3} $\left(\sum \frac{x_i}{\alpha_i} = 0\right)$. *This* S_{n-3} *is the polar space of the quadric as to a unique cubic spread* $(C_3^{(n-2)})$ *which has the same orthic n-space with the same poles. The norm-curve is an apolarity involution curve of this spread and the* $I_{2,\,n-2}$ *of anthorthic sets on it is contained in the* $I_{3,\,n-3}$.

§ 15. *The Perspective Curves of Stahl.*

In an article* entitled "Zur Erzeugung der ebenen rationalen Curven," W. Stahl develops some properties of rational envelopes, $E_m^{(2)}$, perspective to a given rational curve, $R_n^{(2)}$. His mode of treatment applies only to envelopes whose class, m, is less than n. The method here given† is quite general, though it appears that a natural upper limit for m is $2(n-1)$, the class of $R_n^{(2)}$.

Let $u_i = (\lambda_i \tau)^m$ and $x_i = (l_i t)^m$, $i = 1, 2, 3$, be respectively the rational envelope $E_m^{(2)}$ and the rational curve $R_n^{(2)}$ in parametric form. The (m, n) relation between τ and t,

$$(\lambda_1 \tau)^m (l_1 t)^n + (\lambda_2 \tau)^m (l_2 t)^n + (\lambda_3 \tau)^m (l_3 t)^n = 0,$$

expresses that the line τ of E_m and the point t of R_n are incident. The relation expanded in powers of (τt) is

$$(123) \quad \left\{\sum_1^3 (\lambda_i s)^m (l_i s)^n\right\}_{\tau^m t^n} + \frac{m n}{m+n} (\tau t) \left\{\sum_1^3 (\lambda_i l_i)(\lambda_i s)^{m-1} (l_i s)^{n-1}\right\}_{\tau^{m-1} t^{n-1}}$$
$$+ \frac{\binom{m}{2}\binom{n}{2}}{\binom{m+n-1}{2}} (\tau t)^2 \left\{\sum_1^3 (\lambda_i l_i)^2 (\lambda_i s)^{m-2} (l_i s)^{n-2}\right\}_{\tau^{m-2} t^{n-2}} + \dots = 0.$$

---

* *Mathematische Annalen*, Vol. XXXVIII (1891). See also an article by R. Schumacher in the same volume.

† Cf. F. Meyer, *Mathematische Annalen*, Vol. XXIX (1887), p. 447; and H. Brill: "Ueber rationale Curven und Regelflächen," *Münchener Berichte*, 1885.

By definition *E_m and R_n are perspective if like-named line and point are incident.* The necessary and sufficient condition for this is that (τt) be a factor of (123) or that the $(m+n)$-ic, $\sum_1^3 (\lambda_i s)^m (l_i s)^n$, vanish identically. If $R_n^{(2)}$ be given, this imposes $m+n+1$ linear homogeneous conditions on the $3(m+1)$ constants in the forms $(\lambda_i \tau)^m$. Thus there is a system of ∞^{2m-n+1} rational envelopes of class m perspective to a given general $R_n^{(2)}$. The arbitrary constants of the system occur linearly in the parametric equation of the envelope. The multiplicity of the system increases with m, but there is in the general case a lower limit, $m \geq \frac{n-1}{2}$.

Let $\sum_1^3 (\lambda_i s)^m (l_i s)^n \equiv 0$, and let s be a root of $\sum_1^3 (\lambda_i l_i)(\lambda_i s)^{m-1} (l_i s)^{n-1}$. The $(m-1)$-th polar of $\tau = s$ as to this $(m+n-2)$-ic has a root $t = s$, whence the line $\tau = s$ of E_m touches R_n at the point $t = s$. Since the perspective relation is dual, the point $t = s$ of R_n lies on E_m at the contact of $\tau = s$; *i. e.*, E_m and R_n touch with the common element, $\tau = t = s$. The requirement that the $(m+n-2)$-ic have a root s is a linear condition on its coefficients. Thus, as long as $2m-n+1 \leq m+n-2$, or $m < 2(n-1)$, E_m can be uniquely determined by assigning $2m-n+1$ of its $m+n-2$ contacts with R_n. The remaining $2n-m-3$ contacts are then determined. Hence the sets of contacts of E_m's with R_n lie in an involution.

(124) *The rational curve R_n has ∞^{2m-n+1} perspective rational envelopes E_m which, when expressed parametrically, lie in a linear system. Each E_m touches R_n at $m+n-2$ points. If $m < 2(n-1)$, these contacts lie in an involution, $I_{2m-n+1,\, 2n-3-m}$.*

Since the expansion (123) is linear, the parametric sum of two perspective E_m's is a perspective E_m.

Let us ask further under what conditions the contacts of E_m can be assigned so that its class reduces to $m-1$ due to the presence of a common linear factor in the m-ics, $(\lambda_i \tau)^m$. Let $(\lambda_i \tau)^m = (\rho \tau) . (\mu_i \tau)^m$. Since the E_{m-1}, $u_i = (\mu_i \tau)^{m-1}$, is also to be perspective to R_n, $\sum_1^3 (\mu_i s)^{m-1} (l_i s)^n \equiv 0$. Then E_{m-1} is completely determined by giving $2m-n-1$ of the contacts $\sum_1^3 (\mu_i l_i)(\mu_i s)^{m-2} (l_i s)^{n-1} = 0$.

On the other hand, the incidence condition of the degenerate E_m is

$$(\rho \tau) \sum_1^3 (\mu_i \tau)^{m-1} (l_i t)^n = 0.$$

Its expansion is

$$\left\{(\rho s) . \sum_1^3 (\mu_i s)^{m-1} (l_i s)^n\right\}_{\tau^m t^n} + \frac{m n}{m+n} (\tau t) \left\{\frac{m-1}{n} (\rho s) \sum_1^3 (\mu_i l_i) (\mu_i s)^{m-2} (l_i s)^{n-1} \right.$$
$$\left. + \frac{1}{m} \sum_1^3 (\rho l_i) (\mu_i s)^{m-1} (l_i s)^{n-1}\right\}_{\tau^{m-1} t^{n-1}} + \cdots\cdots$$

Since $\sum_1^3 (\mu_i s)^{m-1} (l_i s)^n \equiv 0$, the degenerate E_m must occur in the family of perspective E_m's. By polarizing as to ρ the identity just written and from the usual symbolic identity, two new identities are obtained,

$$\sum_1^3 (m-1) (\mu_i \rho) (\mu_i s)^{m-2} (l_i s)^n + \sum_1^3 n (l_i \rho) (\mu_i s)^{m-1} (l_i s)^{n-1} \equiv 0,$$

$$\sum_1^3 (\mu_i \rho) (\mu_i s)^{m-2} (l_i s)^n - \sum_1^3 (l_i \rho) (\mu_i s)^{m-1} (l_i s)^{n-1} \equiv (\rho s) . \sum_1^3 (\mu_i l_i) (\mu_i s)^{n-2} (l_i s)^{n-1},$$

by means of which the second term of the expansion of the degenerate E_m can be written as

$$\frac{n(m-1)}{n+m-1} (\tau t) \left\{(\rho s) . \sum_1^3 (\mu_i l_i) (\mu_i s)^{m-2} (l_i s)^{n-1}\right\}_{\tau^{m-1} t^{n-1}} .$$

If, therefore, of the $2m-n+1$ arbitrary contacts of E_m, $2m-n$ are chosen which are the contacts of a perspective E_{m-1}, then, however the remaining contact (ρ) is chosen, the E_m so determined is actually the perspective E_{m-1}. Thus the $I_{2m-n+1,\, 2n-m-3}$ of contacts of perspective E_m's contains as neutral sets the $m+n-3$ contacts of perspective E_{m-1}'s.

The E_{2n-3}'s have $3n-5$ contacts which can be chosen at random; the $E_{2(n-2)}$'s have $3(n-2)$ contacts which lie in an $I_{3n-7,\,1}$, whence these contacts form sets of $3(n-2)$ points on R_n whose parameters are apolar to a form, P, of order $3(n-2)$. From the well-known properties of neutral sets we have

(125) *The* $3n-5$ *contacts of an* E_{2n-3} *perspective to the general* R_n *determine the* E_{2n-3}. *There is a set,* P, *of* $3(n-2)$ *points on* R_n *at each of which a single perspective* $E_{2(n-2)}$ *has a contact of order* $3(n-2)$ *with* R_n. *Every perspective* $E_{2(n-2)-k}$ *touches* R_n *at* $3(n-2)-k$ *points whose parameters are an apolar set of the form* P.

To obtain the parametric equation of the perspective E_m, let E_m and R_n be respectively

$$\begin{aligned} u_1 &= \alpha_0 \tau_1^m + \alpha_1 \tau_1^{m-1} \tau_2 + \dots + \alpha_m \tau_2^m, \\ u_2 &= \beta_0 \tau_1^m + \beta_1 \tau_1^{m-1} \tau_2 + \dots + \beta_m \tau_2^m, \\ u_3 &= \gamma_0 \tau_1^m + \gamma_1 \tau_1^{m-1} \tau_2 + \dots + \gamma_m \tau_2^m, \end{aligned}$$

and

$$\begin{aligned} x_1 &= (a t)^n = a_0 t_1^n + a_1 t_1^{n-1} t_2 + a_2 t_1^{n-2} t_2^2 + \dots + a_n t_2^n, \\ x_2 &= (b t)^n = b_0 t_1^n + b_1 t_1^{n-1} t_2 + b_2 t_1^{n-2} t_2^2 + \dots + b_n t_2^n, \\ x_3 &= (c t)^n = c_0 t_1^n + c_1 t_1^{n-1} t_2 + c_2 t_1^{n-2} t_2^2 + \dots + c_n t_2^n. \end{aligned}$$

Let also $(a t)^{n-1} = a_0 t_1^{n-1} + a_1 t_1^{n-2} t_2 + \dots + a_{n-1} t_2^{n-1}$, etc.; let $s^{(1)}, s^{(2)}, \dots, s^{(2m-n+1)}$ be the parameters of the assigned contacts; and let $(u x) = 0$ be the line of E_m with parameter τ. The perspective condition and the given contacts fix the ratios α_i, β_1, γ_i. The additional equation, $(u x) = 0$, permits of their elimination by means of a determinant of order $3(m+1)$. A slight modification of this determinant gives these results:

The perspective point of an R_2 with contact $s^{(1)}$ is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 2 s_2^{(1)} (a s^{(1)}) & 2 s_2^{(1)} (b s^{(1)}) & 2 s_2^{(1)} (c s^{(1)}) \\ \tau_1 x_1 & \tau_1 x_2 & \tau_1 x_3 & \tau_2 x_1 & \tau_2 x_2 & \tau_2 x_3 \end{vmatrix} = 0.$$

The unique perspective point of an R_3 is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 & a_2 & b_2 & c_3 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 \\ \tau_1 x_1 & \tau_1 x_2 & \tau_1 x_3 & \tau_2 x_1 & \tau_2 x_2 & \tau_2 x_3 \end{vmatrix} = 0.$$

The perspective conic of an R_3 with contacts $s^{(1)}$ and $s^{(2)}$ is

$$\begin{vmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & b_1 & c_1 & a_0 & b_0 & c_0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & a_1 & b_1 & c_1 & a_0 & b_0 & c_0 \\ a_3 & b_3 & c_3 & a_2 & b_2 & c_2 & a_1 & b_1 & c_1 \\ 0 & 0 & 0 & a_3 & b_3 & c_3 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 & b_3 & c_3 \\ 0 & 0 & 0 & (a s^{(1)})^3 & (b s^{(1)})^3 & (c s^{(1)})^3 & 2 s_2^{(1)} (a s^{(1)})^2 & 2 s_2^{(1)} (b s^{(1)})^2 & 2 s_2^{(1)} (c s^{(1)})^2 \\ 0 & 0 & 0 & (a s^{(2)})^3 & (b s^{(2)})^3 & (c s^{(2)})^3 & 2 s_2^{(2)} (a s^{(2)})^2 & 2 s_2^{(2)} (b s^{(2)})^2 & 2 s_2^{(2)} (c s^{(2)})^2 \\ \tau_1^2 x_1 & \tau_1^2 x_2 & \tau_1^2 x_3 & \tau_1 \tau_2 x_1 & \tau_1 \tau_2 x_2 & \tau_1 \tau_2 x_3 & \tau_2^2 x_1 & \tau_2^2 x_2 & \tau_2^2 x_3 \end{vmatrix} = 0$$

Let $\{\delta\}$ denote the matrix $\left\|\begin{array}{c} a_0\, b_0\, c_0 \\ \cdots\cdots\cdots \\ a_n\, b_n\, c_n \end{array}\right\|$ of the coefficients; let $\left\|\underline{\{\delta\}\ \{\delta\}\ \{\delta\}\cdots\cdots}\right\|$ denote the matrix formed by placing a series of matrices $\{\delta\}$ side by side, each one line lower than the preceding one, the elements not otherwise provided for being zero; let a matrix like $\| 2s_2(as)^2\ 2s_2(bs)^2\ 2s_2(cs)^2 \|$ be denoted by $2s_2|(as)^2|$, and a matrix like $\|\tau_1^2 x_1\ \tau_1^2 x_2\ \tau_1^2 x_3\|$ be denoted by $\tau_1^2|x|$. In this new notation the three determinants just written are

$$\left|\begin{array}{cc} \{\delta\} & \{\delta\} \\ \hline |0| & 2s_2|(as)| \\ \tau_1|x| & \tau_2|x| \end{array}\right| = 0, \quad \left|\begin{array}{cc} \{\delta\} & \{\delta\} \\ \hline \tau_1|x| & \tau_2|x| \end{array}\right| = 0, \quad \left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^3| & 2s_2^{(1)}|(as^{(1)})^2| \\ |0| & |(as^{(2)})^3| & 2s_2^{(2)}|(as^{(2)})^2| \\ \tau_1^2|x| & \tau_1\tau_2|x| & \tau_2^2|x| \end{array}\right| = 0.$$

The $I_{2,1}$ of contacts of perspective conics of an R_3 is

$$\left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^3| & 2s_2^{(1)}|(as^{(1)})^2| \\ |0| & |(as^{(2)})^3| & 2s_2^{(2)}|(as^{(2)})^2| \\ |0| & |(as^{(3)})^3| & 2s_2^{(3)}|(as^{(3)})^2| \end{array}\right| = 0.$$

Similarly the equations of the perspective conics, cubics, quartics and quintics of an R_4 are

$$\left|\begin{array}{ccc} \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |(as^{(1)})^4| & 2s_2^{(1)}|(as^{(1)})^3| \\ \tau_1^2|x| & \tau_1\tau_2|x| & \tau_2^2|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{cccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)}|(as^{(1)})^4| & 2s_2^{(1)}|(as^{(1)})^4| & 3s_2^{(1)^2}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^3|x| & \tau_1^2\tau_2|x| & \tau_1\tau_2^2|x| & \tau_2^3|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{ccccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)^2}|(as^{(1)})^4| & 2s_1^{(1)}s_2^{(1)}|(as^{(1)})^4| & 3s_2^{(1)^2}|(as^{(1)})^4| & 4s_2^{(1)^3}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^4|x| & \tau_1^3\tau_2|x| & \tau_1^2\tau_2^2|x| & \tau_1\tau_2^3|x| & \tau_2^4|x| \end{array}\right| = 0,$$

$$\left|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & s_1^{(1)^3}|(as^{(1)})^4| & 2s_1^{(1)^2}s_2^{(1)}|(as^{(1)})^4| & 3s_1^{(1)}s_2^{(1)^2}|(as^{(1)})^4| & 4s_2^{(1)^3}|(as^{(1)})^4| & 5s_2^{(1)^4}|(as^{(1)})^3| \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \tau_1^5|x| & \tau_1^4\tau_2|x| & \tau_1^3\tau_2^2|x| & \tau_1^2\tau_2^3|x| & \tau_1\tau_2^4|x| & \tau_2^5|x| \end{array}\right| = 0,$$

where in each case a sufficient number of contacts, $s^{(1)}, s^{(2)}, \ldots$, must be given to complete the determinant. In each case also the involution of contacts is obtained by assigning an additional contact and dropping the condition $(ux) = 0$; *i. e*, the last row is replaced by one like those that immediately precede it.

The above examples suffice to indicate the equation of the general E_m perspective to R_n, $m < 2(n-1)$. We have further that

(126) *The vanishing of the matrix* $\| \{\delta\} \{\delta\} \ldots \{\delta\} \|$ *formed from* ρ *matrices* $\{\delta\}$ *gives the single condition that an* $R_{2\rho}$, *or the conditions that an* $R_{2\rho+\sigma}$, *have a perspective* $E_{\rho-1}$.

The degree of the contact involutions in the combinant coefficients, δ_i; *i. e.*, in the three-row minors of $\{\delta\}$, is seen to be $m+1$, whence

(127) *The degree of the form* P *in* δ_i *is* $2n-3$. *Its apolar forms of order* $3(n-2)-j$ *lie in an involution of degree* $2n-3-j$ *in* δ_i.

Returning to the original incidence condition (123), let the coefficients of both $(\tau t)^0$ and $(\tau t)^1$ vanish. Then E_m and R_n will be called "doubly perspective." They must touch at every point of R_n and are merely dual forms of the same locus. If then, we require that R_n have a doubly perspective $E_{2(n-1)-1}$, this imposes the condition that R_n have a cusp. The equations of condition number $(m+n+1)+(m+n-1) = 6(n-1)$. The $6(n-1)$ coefficients of $E_{2(n-1)-1}$ can be eliminated from them by means of order $6(n-1)$, which is the cusp condition of degree $2(n-1)$ in δ_i. For the R_3 this determinant is easily manipulated into the form

$$\left| \begin{array}{cccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline |0| & |a_0| & |0| & |0| \\ |0| & |a_1| & 2|a_0| & |0| \\ |0| & |a_2| & 2|a_1| & 3|a_0| \\ |0| & |a_3| & 2|a_2| & 3|a_1| \\ |0| & |0| & 2|a_3| & 3|a_2| \end{array} \right| = 0,$$

which again can be indicated more briefly by

$$\left| \begin{array}{cccc} \{\delta\} & \{\delta\} & \{\delta\} & \{\delta\} \\ \hline & \{\delta\} & 2\{\delta\} & 3\{\delta_3\} \\ \hline \end{array} \right| = 0,$$

where $\{\delta_3\}$ denotes the matrix $\{\delta\}$ with the row $|a_3|$ deleted. In the general case we find that

(128) *The cusp condition of* R_n *is the vanishing of the determinant*

$$\left|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \dots\dots & \{\delta\} & \{\delta\} \\ \hline & \{\delta\} & 2\{\delta\} & \dots\dots & (2n-4)\{\delta\} & (2n-3)\{\delta_n\} \\ \hline \end{array}\right|$$

of degree $2(n-1)$ *in* δ_i.

This is included in the more general theorem:

(129) *If* R_n *be actually of order n (i. e., if the n-ics* $(l_i t)^n$ *have no common factor), the condition that it have h cusps or their equivalent is the vanishing of the matrix*

$$\left\|\begin{array}{cccccc} \{\delta\} & \{\delta\} & \{\delta\} & \dots\dots & \{\delta\} & \{\delta\} \\ \hline & \{\delta\} & 2\{\delta\} & \dots\dots & (2n-4-h)\{\delta\} & (2n-3-h)\{\delta\} \\ \hline \end{array}\right\|,$$

which expresses that it has a doubly perspective $E_{2(n-1)-h}$.

The existence of doubly perspective envelopes of class $m \gtrless 2(n-1)$ explains why the contacts alone are no longer sufficient to determine the envelope. Thus if R_n in lines is $u_i = (\mu_i \tau)^{2(n-1)}$ and E_m is a perspective envelope of class $m = 2(n-1) + k$, then the ∞^{k+1} envelopes $u_i = (\lambda_i \tau)^m + (\rho\tau)^k (\mu_i \tau)^{2(n-1)}$ obtained by varying the coefficients of $(\rho\tau)^k$ are all perspective and have the same contacts as E_m.

If, from the incidence condition (123), the factor (τt) be removed, the resulting $(m-1, n-1)$ relation between τ and t has a certain number of involutive pairs which are the pairs of parameters of the double points of R_n and the double tangents of E_m. They are found by eliminating τ from the two equations $(\lambda\tau)^{m-1}(lt)^{n-1} = 0$ and $(\lambda t)^{m-1}(l\tau)^{n-1}$. The resultant is of order $(m-1)^2 + (n-1)^2$ in t and gives the $(m-1)(m-2) + (n-1)(n-2)$ double point and double tangent parameters and the $m + n - 2$ contact parameters which are degenerate involutive pairs. Since the double point parameters always satisfy the $(m-1, n-1)$ relation, if R_n has a cusp, the cusp parameter satisfies the relation and counts as a contact; whence

(130) *If* E_m *and* R_n *are perspective,* E_m *passes through all the cusps of* R_n *and* R_n *touches all the flex-tangents of* E_m.

The form P of order $3(n-2)$ and of degree $2n-3$ in δ_i has apolar forms of order $3(n-2)-j$ which lie in an involution of degree $j+1$ in the coefficient of P and therefore of degree $(2n-3)(j+1)$ in δ_i. According to (127) the actual degree of this involution is $2n-3-j$. The outstanding factor of degree $2(n-1)j$ must be the j-th power of the cusp condition. For, if R_n has a cusp, the cusp is a contact of all perspective E_m's. It must therefore be the contact of those envelopes of class $2(n-2)$ whose contacts all coincide at the points, $P=0$ (125). Hence $P=0$ is the cusp parameter repeated $3(n-2)$ times, and the contact involutions when formed directly from P must vanish identically. These involutions are now (disregarding the cusp contact) apolar forms of a form P_1 of order $3(n-2)-2$ and of degree $(2n-3)-1$ in δ_i. Hence

(131) *If R_n has a cusp, the covariant P is the cusp parameter repeated $3(n-2)$ times. If R_n has two cusps, P vanishes. If R_n has h cusps, the contact involutions of perspective envelopes are (omitting the improper cusp contacts) the involutions of forms apolar to a covariant P_h of order $3(n-2)-2h$ and of degree $(2n-3)-h$ in δ_i.*

The following fairly evident theorem furnishes the connection with the preceding § 14.

(132) *The $(m-1, n-1)$ relation, $(\lambda\tau)^{m-1}(lt)^{n-1}=0$, can be obtained from perspective E_m and R_n if $m-2$ linearly independent forms $(\mu^k\tau)^m$ exist such that $(\mu_k t)(\mu_k\lambda)^{m-1}(lt)^{n-1}\equiv 0$, or also if $n-2$ linearly independent forms $(m_k t)^n$ exist such that $(\lambda\tau)^{m-1}(l m_k)^{n-1}(m_k\tau)\equiv 0$. From a relation satisfying the given conditions E_m and R_n can be thus constructed: With regard to a norm-curve N_m in S_m, the $m-2$ forms, $(\mu_k\tau)^m$, are represented by $m-2$ S_{m-1}'s which meet in a plane E. The S_{m-1}'s, τ, of N_m cut E in the lines, τ, of an E_m. If τ and t satisfy the given relation, the two S_{m-1}'s of N_m, τ and t, meet on E in the point t of a rational R_n perspective to E_m. In particular any $(1, n-1)$ relation determines an R_n referred to its perspective conic.*

(133) *The two classes of $R_n^{(2)}$'s for whose apolarity cubic spreads the involution curves were determined in § 14 are, respectively, the $R_n^{(2)}$ with a perspective point (an $(n-1)$-fold point), and the $R_n^{(2)}$ with a perspective conic.*

The foregoing has comparatively little in common with the work of Stahl. Before leaving the subject of perspective curves it may be worth while to develop some of his results in a much simpler way.

The equation of a line through the points t_1 and t_2 of the given R_n is

$$(134)\quad \Gamma=\begin{vmatrix}(l_1t_1)^n & (l_2t_1)^n & (l_3t_1)^n\\ (l_1t_2)^n & (l_2t_2)^n & (l_3t_2)^n\\ x_1 & x_2 & x_3\end{vmatrix}\frac{1}{(t_1t_2)}$$

$$\equiv(\alpha x)(a_1t_1)^{n-1}(a_2t_2)^{n-1}=(\alpha x)(a_1t_2)^{n-1}(a_2t_1)^{n-1}=0.$$

If, in $\Gamma = 0$, t_1 be the parameter of a fixed point p_1 of R_n and t_2 that of a variable point p_2, then, as t_2 varies, the line Γ describes $n-1$ times the fixed point p_1. Thus $\Gamma = 0$ is a parametric equation of the degenerate envelope p_1^{n-1}. Moreover, this envelope is perspective to R_n since the line $\tau = t_2$ contains the point $t = t_2$ of R_n. As p_1 runs over R_n, ∞^1 such degenerate envelopes are obtained. By forming their general linear combination, Stahl's general perspective E_{n-1} is obtained. The process of forming such a sum is merely that of forming the apolarity invariant of a general $(n-1)$-ic, $(b\,t)^{n-1}$, with Γ of order $n-1$ in t, whence, in terms of (134), the equation of the general perspective E_{n-1} is

$$(135)\qquad (\alpha\, x)\,(a_1 b)^{n-1}\,(a_2 \tau)^{n-1} = 0.$$

This is the equation of a line which, for given τ, passes through the point $t = \tau$ of R_n and cuts R_n again in $n-1$ points whose parameters are apolar to $(b\,t)^{n-1}$. Hence

(136) *The lines which cut R_n in $n-1$ points whose parameters are apolar to an arbitrarily chosen $(n-1)$-ic, $(b\,t)^{n-1}$, envelop a perspective E_{n-1}. If, however, $(b\,t)^{n-1}$ is the polar as to t_1 of a set F^n of the fundamental involution, E_{n-1} is made up of the lines through t_1 on R_n and a perspective envelope E_{n-2} which is independent of the choice of t_1; if $(b\,t)^{n-1}$ is the polar as to t_1 and t_2 of a form F^{n+1} whose first polars all are sets of the fundamental involution, E_{n-1} is made up of lines on t_1, lines on t_2, and a perspective E_{n-2} which is independent of t_1 and t_2, i. e., which depends only on F^{n+1}; etc.*

For it is quite clear that the lines on t_1 (on t_1 or t_2, etc.) cut R_n in $(n-1)$-ics apolar to $(b\,t)^{n-1}$; but there is also a further system of ∞^1 lines which cut R_n in $(n-1)$-ics which are canonizant sets of F^n (F^{n+1}, etc.); these $(n-1)$-ics are apolar to $(bt)^{n-1}$ whatever be t_1 (t_1 and t_2, etc.).

Returning to the degenerate E_{n-1}, a point of R_n repeated $n-1$ times, its contacts are the point itself and the $2(n-2)$ contacts of tangents from the point. If the point is a double point, t_1, t_1' on R_n, then another factor $(t_1' t_2)$ separates from Γ. The E_{n-1} becomes an E_{n-2} whose contacts are at the double point and at the $2(n-3)$ contacts of tangents from the double point. Since these contacts are apolar to P, we have that

(137) *The parameters of a node and the $2(n-3)$ parameters of tangents from the node are apolar to the covariant P of order $3(n-2)$. Thus only $n-2$ of the $\frac{1}{2}(n-1)(n-2)$ such $2(n-2)$-ics are linearly independent.*

§16. *The Gordan Expansion of the Form $H_{2,3}$.*

The expansion of $H_{2,3} = (a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3$ is a sum of terms each of which is a product of identical covariants, $(x_i x_k)$, and of a polarized form. The polarized form must be a linear covariant of $H_{2,3}$. Because of the equivalence of the symbols, a_1, a_2, a_3 in $H_{2,3}$, its linear covariants must be the same symbolically as the covariants of the third degree of a binary cubic,

$$(a_1 x)^3 = (a_2 x)^3 = (a_3 x)^3.$$

These are the cube of the cubic, the product of the cubic and its Hessian, and the cubicovariant. Hence the required linear covariants of $H_{2,3}$ are:

$$\begin{aligned}(\pi x)^9 &= (a_1 x)^3 (a_2 x)^3 (a_3 x)^3,\\ (\rho x)^5 &= (a_1 a_2)^2 (a_1 x)(a_2 x)(a_3 x)^3,\\ (\sigma x)^3 &= (a_1 a_2)^2 (a_3 a_1)(a_2 x)(a_3 x)^2.\end{aligned}$$

Some products which occur in the expansion are:

$$\begin{aligned}p_1 &= (x_2 x_3)^2 (\rho x_2)(\rho x_3)(\rho x_1)^3,\\ p_2 &= (x_3 x_1)^2 (\rho x_3)(\rho x_1)(\rho x_2)^3,\\ p_3 &= (x_1 x_2)^2 (\rho x_1)(\rho x_2)(\rho x_3)^3,\end{aligned}$$

$$\begin{aligned}r_1 &= (x_2 x_3)^2 (x_2 x_1) \,.\, (\sigma x_3)(\sigma x_1)^2, & s_1 &= (x_2 x_3)^2 (x_3 x_1) \,.\, (\sigma x_2)(\sigma x_1)^2,\\ r_2 &= (x_3 x_1)^2 (x_3 x_2) \,.\, (\sigma x_1)(\sigma x_2)^2, & s_2 &= (x_3 x_1)^2 (x_1 x_2) \,.\, (\sigma x_3)(\sigma x_2)^2,\\ r_3 &= (x_1 x_2)^2 (x_1 x_3) \,.\, (\sigma x_2)(\sigma x_2)^2, & s_3 &= (x_1 x_2)^2 (x_2 x_3) \,.\, (\sigma x_1)(\sigma x_3)^2.\end{aligned}$$

In terms of these the expansion is:

$$\begin{aligned}(138)\quad (a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3 = (\pi x_1)^3 (\pi x_2)^3 (\pi x_3)^3 &+ \tfrac{9}{16} (p_1 + p_2 + p_3)\\ &- \tfrac{9}{35} (r_1 + r_2 + r_3 + s_1 + s_2 + s_3).\end{aligned}$$

If $H_{2,3}$ be interpreted as the apolarity equation of a cubic surface, C_3, with regard to an N_3, then $(\pi x)^9 = 0$ is the equation of the parameters on N_3 of the meets of N_3 and C_3. If $(\pi x)^9 \equiv 0$, then N_3 lies on C_3. In this case a tangent plane of C_3 at a point t on N_3 touches N_3 at t and meets it again at τ. The relation between t and τ is $(a_1 \tau)^3 (a_2 t)^3 (a_3 t)^3 = 0$, which now becomes

$$(139)\qquad 9(\tau t)^2 \{ \tfrac{1}{8} (\rho \tau)(\rho t)^4 - \tfrac{2}{35} (\tau t)(\sigma t)^3 \} = 0.$$

τ and t coincide at a point where N_3 touches an asymptotic curve of C_3, say at an "asymptotic point" of N_3. Thus if $(\pi x)^9 \equiv 0$, $(\rho x)^5 = 0$ determines the five asymptotic points on N_3. If also $(\rho x)^5 \equiv 0$, N_3 is an asymptotic curve of C_3 which has double points at the three points, $(\sigma x)^3 = 0$, of N_3.

Denote the cubic surface by D_3 if it contains N_3. Let the apolarity form of D_3 be $(d_1 x_1)^3 (d_2 x_2)^3 (d_3 x_3)^3$, whence $(d_1 \tau)^3 (d_2 t)^3 (d_3 t)^3 = (\tau t)^2 \,.\, (\lambda \tau)(l t)^4$. By comparison with (139) one sees that the (1, 4) form $(\lambda\tau)(lt)^4$ can replace the apolarity form since it determines $(\rho x)^5$ and $(\sigma x)^3$. It is also evident that

(140) *The condition that D_3 have one, two, three, or four double points on N_3 is that from $(\lambda\tau)(lt)^4$ there shall separate respectively one, two, three, or four factors linear in t. If both $(\lambda\tau)(lt)^4 = 0$ and $(\lambda t)(l\tau)^4 = 0$, the join of t and τ is one of the six chords of N_3 on D_3. If $(\lambda\tau)(lt)^4$ contains a factor cubic in t, and if the other factor is symmetric in t and τ, then D_3 is the product of a quadric on N_3 and a plane which cuts N_3 in the points given by the cubic factor.*

Let us ask further under what conditions a quadric cone on N_3 with vertex t_1 shall touch D_3 along N_3. Necessarily then, if $\lambda_1 (t_1 x)^3 + \lambda_2 (t_2 x)^3$ [which, as t_2 runs along N_3, is for variable $\lambda_1 : \lambda_2$ a generator of the cone] be substituted in $(d_1 x_1)^3 (d_2 x_2)^3 (d_3 x_3)^3$, the term in $\lambda_1 \lambda_2^2$ vanishes, *i. e.*,

$$(d_1 t_1)^3 (d_2 t_2)^3 (d_3 t_2)^3 = (t_1 t_2)^2 \,.\, (\lambda t_1)(l t_2)^4 = 0.$$

Since $(\lambda t_1)(l t_2)^4$ vanishes for all values of t_2, the form $(\lambda\tau)(lt)^4$ must be $(\tau t_1) \,.\, (q t_2)^4$, whence

(141) *In order that a quadric cone on N_3 shall touch D_3 along N_3 it is necessary that D_3 shall have four double points on N_3. Conversely if D_3 has four double points and if N_3 lies on D_3 and its four double points, then the tangent planes of D_3 at points on N_3 meet N_3 again in a unique point, the vertex of a quadric cone on N_3 which touches D_3 along N_3.**

§ 17. *The Form $I_{2,3}$ and the Curve $R_5^{(2)}$.*

Let the $R_5^{(2)}$, $x_i = (l_i t)^5$, determine the $I_{2,3}$, $(a_1 x_1)^3 (a_2 x_2)^3 (a_3 x_3)^3 = 0$. The three quintics, $(m_i t)^5$, apolar to $(l_i t)^5$ determine the "apolar" or "fundamental" involution, $\bar{I}_{2,3}$. Thus an $R_5^{(2)}$ projectively determines an $\bar{R}_5^{(2)}$. Each involution is easily expressed in terms of the sets of the other; thus

$$I_{2,3} = (m_1 m_2)(m_1 m_3)(m_2 m_3) \Pi (m_i x_k),$$
$$\bar{I}_{2,3} = (l_1 l_2)(l_1 l_3)(l_2 l_3) \Pi (l_i x_k), \qquad i, k = 1, 2, 3.$$

---

*If D_3 be mapped on a plane by means of cubic curves on the six points of a 4-line, (141) can be stated thus: Cubic curves on the six points of a 4-line with double points on a line l all meet l again in a single point p. It can be shown from other considerations that p is the point of contact of a conic which touches the 4-line and l.

Every covariant of one involution is a covariant of the other; in particular the flex-equations of $R_5^{(2)}$ and $\bar{R}_5^{(2)}$,

$$(m_1 m_2)(m_1 m_3)(m_2 m_3)\Pi(m_i x)^3 = 0$$

and

$$(l_1 l_2)(l_1 l_3)(l_2 l_3)\Pi(l_i x)^3 = 0,$$

are the same. The linear covariants,

$$(l_2 l_3)^5(l_1 x)^5 + (l_3 l_1)^5(l_2 x)^5 + (l_1 l_2)^5(l_3 x)^5$$

and

$$(m_2 m_3)^5(m_1 x)^5 + (m_3 m_1)^5(m_2 x)^5 + (m_1 m_2)^5(m_3 x)^5,$$

are the same; they are the set common to both involutions.

If $I_{2,3}$ be expanded in a Gordan series (138), the various terms contain polars of $(\pi x)^9$, $(\rho x)^5$, and $(\sigma x)^3$. Since these forms are the only linear covariants of $I_{2,3}$, $(\pi x)^9 = 0$ must be the flex-equation of $R_5^{(2)}$ and $(\rho x)^5 = 0$ must be the set common to $I_{2,3}$ and $\bar{I}_{2,3}$. If $\bar{I}_{2,3}$ be similarly expanded, its terms also must contain polars of the same forms. If the factors of proportionality be chosen so that $(\pi x)^9$ is the same in both expansions, the other forms can differ only in sign. But the identical vanishing of $(\rho x)^5$ is the condition that $I_{2,3}$ and $\bar{I}_{2,3}$ coincide. Hence

(142) *If in the Gordan expansion* (138) *of* $I_{2,3}$, *the sign of* $(\rho x)^5$ *be changed, the resulting expansion is that of the involution* $\bar{I}_{2,3}$.

Let h be a quadric on N_3 whose equation is $(x_1 x_2)^2(hx_1)(hx_2) = 0$; and let the plane equation of a point quadric, $(a_2 x_2)^3(a_3 x_3)^3 = 0$, be $(\alpha_2 x_2)^3(\alpha_3 x_3)^3 = 0$, (102). Then the associated cubic surface, C_h, (110), of the quadric h has the apolarity equation

$$C_h = (a_1 x_1)^3(a_1' x_2)^3(a_1'' x_3)^3(\alpha_2 \alpha_3)^2(\alpha_2 h)(\alpha_3 h) = 0.$$

We have seen (p. 342) that C_h must contain N_3, whence

$$(a_1 x)^3(a_1' x)^3(a_1'' x)^3(\alpha_2 \alpha_3)^2(\alpha_2 h)(\alpha_3 h) = 0.$$

But if a cubic surface contains N_3 its apolarity equation can be replaced by a (1, 4) form (p. 360); *i. e.*,

$$[C_h]_{x_2=x_3} = (x_1 x_2)^2 \,.\, (p x_1)(q x_2)^4(r h)^2.$$

Again (p. 342), the intersection of h and C_h is N_3 and a residual cubic curve which meets N_3 in the five points of one of the 16 pentahedra of (109). Every quadric on N_3 cuts out such a residual cubic except the cones on N_3. If h be a cone on N_3 with vertex τ, the residual cubic must coincide with N_3, and h must touch C_h along N_3. This requires (141) that the (1, 4) form $[C_h]_{x_2=x_3}$ degenerate into a linear form which locates the vertex of the cone and a quartic form which locates four double points of C_h; *i. e.*, the form $(px_1)(qx_2)^4(r\tau)^2$ is merely the product $(\tau x_1) \cdot (\lambda\tau)\,(lx_2)^4$. As the vertex τ runs over N_3, the four double points of the associated cubic surface also run over N_3 and a unique (1, 4) form, $(\lambda\tau)\,(lt)^4$, associated with $I_{2,3}$, is determined, which is of the third degree in the coefficients of $I_{2,3}$.

In order to identify the form let r_1, r_2 be the parameters on N_3 of a pair of corresponding points of the Hessian of $C_3 = a_x^3$; let Q_{r_1} be the cone on N_3 with vertex r_1, and let P_{r_1} be the polar quadric of r_2 as to C_3 which is also a cone with vertex at r_1. Since there are ∞^1 triads, S_1, S_2, S_3, on N_3, which with r_1 and r_2 make up $A_{3,3}^5$'s of C_3, then $a_{r_2} a_{s_i} a_{s_k} = 0$. Since $a_{r_2} a_x^2$ is P_{r_1}, there are ∞^1 3-lines on Q_{r_1} self-conjugate as to P_{r_1}; *i. e.*, $(P_{r_1} P_{r_1} Q_{r_1} u)^2 \equiv 0$. Operating with this on a_x^3, we have

$$(P_{r_1} P_{r_1} Q_{r_1} a)^2 a_x = a_x b_{r_2} c_{r_2} (abc\, Q_{r_1})^2 \equiv 0.$$

This expresses that the associated cubic surface of the cone Q_{r_1} has a double point at r_2. Similarly the associated cubic surface of Q_{r_2} has a double point at r_1, whence r_1 and r_2 make up an involutive pair of the (1, 4) form just found. These results can be summarized thus:

(143) *The general* $R_5^{(2)}$ *has a unique perspective conic,* K, *whose incidence condition is a general* (1, 4) *form,* $(\lambda\tau)\,(lt)^4 = 0$. *In the representation of* $R_5^{(2)}$ *on* N_3 *this form has the following three interpretations:*

(a) *The apolarity involution curve* N_3 *of* C_3 *contains* 5 *poles as to* C_3 *of the Sylvester* 5-*plane of* C_3. *The* 5-*point and* 5-*plane are pole and polar as to a quadric* Q. *The polar quadric as to* C_3 *of a point* τ *has in common with* Q *a self-polar* 4-*point* t *on* N_3.

(b) *The associated cubic surface as to* C_3 *of a quadric cone on* N_3 *with vertex* τ *has four double points* t *on* N_3.

(c) N_3 *cuts the Hessian of* C_3 *in six pairs of corresponding points whose joins,* L_i, *belong to a double six of a cubic surface* D_3 *on* N_3. *The tangent plane of* D_3 *at a point* t *cuts* N_3 *again at a point* τ.

The (1, 4) *form is of the third degree in the coefficients of* $I_{2,3}$. *It is obtained from* $(a_1 x_1)^3 (a_1' x_2)^3 (a_1'' x_3)^3 (\alpha_2 \alpha_3)^2 (\alpha_2 h)\,(\alpha_3 h)$ *by putting* $x_2 = x_3 = t$ *and* $(hx)^2 = (\tau x)^2$, *and then removing the factors* $(x_1 t)^2 (x_1 \tau)$. *The* $I_{2,3}$ *is also of the third degree in the coefficients of the* (1, 4) *form. The extraneous invariant is the triple point condition.*

We have already remarked, that the (1, 4) form serves to replace the apolarity equation of a cubic surface on N_3. Thus the apolarity equation of D_3, a simultaneous covariant of C_3 and N_3, is known in terms of $(\lambda\tau)(lt)^4$.

(144) *The invariant theory of the general* $R_5^{(2)}$ *coincides with that of a general* (1, 4) *binary form.*

It is known that if either a given involution of n things or its apolar involution contains only one or only two independent forms, then the covariants of the involution can be expressed as covariants of a *single* general binary form. This includes the $I_{0,n}$ or $I_{n-1,1}$, in which case the single form is an n-ic; and the $I_{1,n-1}$ or $I_{n-2,2}$, in which case the single form is a $2(n-1)$-ic. The $I_{2,3}$ is the simplest involution not included in these cases and (144) gives the analogous theorem for the $I_{2,3}$. The advantage in discussing a form whose coefficients are subject to no relations is manifest.

If the cubic surface D_3 be mapped on a plane E so that the six lines L_i become points p_i, then N_3 maps into the $R_5^{(2)}$ with nodes at p_i. Quadrics on N_3 meet D_3 in residual cubic curves which map into the lines of E; in particular, the residual cubics which map into the lines of K are on the cones of N_3. Cubic surfaces on N_3 meet D_3 in residual sextics which map into quartic curves on the points p_i; in particular, plane sections of D_3 map into the adjoint cubics of $R_5^{(2)}$. The quartic developable of N_3 meets D_3 in a residual rational sextic which maps into K. Let x be a point of D_3, which is on the chord c_x through the points q_1 and q_2 of N_3. Let x map into a point y of E, and q_1 and q_2 into points q_1 and q_2 on $R_5^{(2)}$. Plane sections of D_3 on c_x map into the pencil of adjoint cubics of $R_5^{(2)}$ on q_1, q_2, and y. Since c_x is contained on the cones of N_3 with parameters q_1 and q_2, y is the meet of the tangents of K with parameters q_1 and q_2. Thus we have obtained Stahl's theorem: The pencil of adjoint cubics on two points of $R_5^{(2)}$ meet in a further point which is also the meet of the corresponding two tangents of K. Stahl's very pretty corollary—that a conic on five double points of $R_5^{(2)}$ contains ∞^1 3-points whose joins envelop K—can be applied to express the conditions on the six quadratics which determine the nodes of $R_5^{(2)}$. Only four of these can be chosen at random, leaving one degree of freedom for the other two. With reference to a norm-conic K, let the four given quadratics determine four points, p_1, p_2, p_3, p_4. Through these points there pass two conics C' and C'' which contain the vertices of triangles circumscribed to K. A tangent of K meets C' in two points whose further tangents to K meet at a point p_5 on C'. The same tangent meets C'' in two points whose further tangents to K meet at a point p_6 on C''. The points $p_1, \ldots, p_6$ determine on K the nodal parameters of an $R_5^{(2)}$; or there is an $R_5^{(2)}$ with double points $p_1, \ldots, p_6$ and with the perspective conic K.

The analogous construction for the six lines L_i of a double six is: Given four chords, L_1, L_2, L_3, L_4, of N_3, no three of which are on a quadric that contains N_3; if C' and C'' are the two transversals of $L_1, \ldots, L_4$, the planes through C' and C'', respectively, and any point of N_3 cut N_3 again in two further pairs of points whose joins are the lines L_5 and L_6.

BALTIMORE, *January* 25, 1910.